Controllability of NLS in the vicinity of solitary wave solutions

H. Lange (Cologne)
H. Teismann (Acadia)

February 2, 2008

Abstract

Local exact controllability of the 1D NLS (subject to zero boundary conditions) with distributed control is shown to hold in a H^1–neighbourhood of the nonlinear ground state. The *Hilbert Uniqueness Method (HUM)*, due to J.-L. Lions, is applied to the linear control problem that arises by linearization around the ground state. The application of HUM crucially depends on the spectral properties of the linearized NLS operator which are given in detail.

1 Introduction

The control properties of many PDEs arising in physics and engineering have been studied extensively. Those investigations include exact and/or optimal controllability for the (linear and nonlinear) heat, wave, beam and plate equations as well as equations of elasticity and Navier Stokes equations, to mention just some of the most prominent examples. For Schrödinger equations, however, the control theory is markedly less developed. For a brief survey on control results for (linear and nonlinear) Schrödinger equations, see e.g. [7]. The purpose of this paper is to establish local exact controllability for the 1D cubic nonlinear Schrödinger equation (subject to zero boundary conditions) with distributed control. Specifically, we will consider

the following control problem.

$$
\begin{aligned}
&\text{(1a)} & iy_t &= -y_{xx} + f(|y|^2)y + g_{\Omega'}(x)u(t,x), \quad (x \in (0,1),\, t \in (0,T))\\
&\text{(1b)} & y(t,0) &= y(t,1) = 0 \quad (t \in [0,T])\\
&\text{(1c)} & y(0,x) &= y_0(x) \quad (x \in (0,1))\\
&\text{(1d)} & y(T,x) &= y_1(x) \quad (x \in (0,1))
\end{aligned}
$$

where $g_{\Omega'}(x)$ denotes the indicator function for some fixed, possibly small, open subinterval $\Omega' \subset \Omega := (0,1)$ which represents the spatial region in which the control is applied. Given a fixed control "horizon" $T > 0$ and initial and target states y_0 and y_1, the objective is to construct a control function $u = u(t,x)$ that will steer the state y from y_0 to y_1, i.e. the unique solution $y = y(t,x)$ of (1a)-(1c) is to satisfy (1d). The control problem (1a)-(1d) was posed in [17, 16]. A small-data controllability result for periodic boundary conditions is contained in [6].
In the remainder of this paper, we will concentrate on the special case of the focusing cubic nonlinear Schrödinger equation (NLS), i.e.

$$f(s) = -s.$$

More general nonlinearities could be treated, but this is not our main interest here. So we restrict ourselves to the prototypical cubic nonlinearity. Our main result states that the control problem (1a)-(1d) is soluble locally within an H^1–neighbourhood of the (ground-state) *solitary-wave* (or *soliton*) solution $\varphi_\mu(t,x) = e^{i\mu t}\phi_\mu(x)$, if the control time $T > 0$ is sufficiently large. Here $\phi_\mu(x)$ denotes the (nonlinear) ground state[1], i.e. the (real and) positive solution of the boundary value problem

$$
\begin{aligned}
&\text{(2a)} & -\phi'' &+ \mu\phi - \phi^3 = 0 \quad (x \in \Omega)\\
&\text{(2b)} & \phi(0) &= \phi(1) = 0.
\end{aligned}
$$

which is known to exist and to be unique (see Section A.1). Our main results reads.

Theorem 1 *Let $\Omega' \subset (0,1)$, $\mu > 0$ be given and let ϕ_μ denote the ground state. Then there exist $T > 0$ and $\delta > 0$ such that, for any y_0, $y_1 \in H_0^1(0,1) := \{v \in H^1(0,1) \mid v(0) = v(1) = 0\}$ satisfying*

$$\|y_0 - \phi_\mu\|_{H^1} < \delta \quad \textit{and} \quad \|y_1 - e^{i\mu T}\phi_\mu\|_{H^1} < \delta,$$

[1] Often the time-dependent solution $\varphi_\mu(t,x)$ is referred to as the ground state.

the control problem (1a)-(1d) has a solution $u = u(t,x)$, i.e. there exists a control function $u \in C([0,T]; H_0^1(0,1))$ such that the unique solution $y \in C([0,T]; H_0^1(0,1))$ of (1a)-(1c) satisfies (1d).

The theorem will be proved by applying the implicit function theorem (IFT) to the nonlinear map $\Psi : [H_0^1(0,1)]^2 \times C([0,T]; H_0^1(0,1)) \to H_0^1(0,1)$, defined by

$$\Psi(y_0, y_1, u) := y(T; y_0, u) - y_1, \tag{3}$$

where $t \mapsto y(t; y_0, u) \in C([0,T]; H_0^1(0,1))$ denotes the unique solution of (1a)-(1c). To be able to apply the IFT, it will be verified that the linearization $\partial_u \Psi(\phi_\mu, e^{i\mu T}\phi_\mu, 0) : C([0,T]; H_0^1(0,1)) \to H_0^1(0,1)$ of Ψ at the point $(\phi_\mu, e^{i\mu T}\phi_\mu, 0)$ exists as a bounded map and possesses a bounded inverse. This amounts to showing that the linear PDE that arises by linearizing NLS around the stationary solution φ_μ is exactly controllable; this, in turn, is done by employing the *Hilbert Uniqueness Method* (HUM) due to J.-L. Lions [9]. The main difficulty in the application of HUM stems from the lack of selfadjointness of the linearized operator, which makes determining its spectral properties more intricate. The analysis reveals that most spectral properties known to hold for the linearized NLS operator in the *whole-space* case [10, 4], carry over to the *zero-boundary* case considered in this paper (see Section 3.1.1)[2]. This fact is of independent interest, but it is, to the best of our knowledge, not available in the literature.

This paper is organized as follows. In Section 2 we formulate the linear control problem that arises by linearization around the ground state (system (9a)-(9d)) and state its solvability (Th. 2). We also show how Theorem 1 is derived from this controllability result (Section 2.2). Section 3 contains the proof of Th. 2. The proof is based on HUM and, thus, hinges on the "observability estimates" that are proved in Sections 3.1 (L^2-estimate) and 3.2 (H^1-estimate). The all-important spectral properties needed in these proofs are listed in Section 3.1.1 and proved in the Appendix (B.3). The Appendix also contains information on explicit solution formulas for (2a),(2b) (Section A.1), a variational description of the ground state (A.2), and asymptotic formulas for the eigenvalues and eigenfunctions (B.2). A number of open problems are listed in Section 4.

[2]Obviously, this statement is not meant to be applied to those properties which crucially depend on the fact that the spatial domain $\Omega = (0,1)$ is bounded, such as the absence of a continuous part of the spectrum.

2 Linearization and proof of Theorem 1

Let $T > 0$ be such that property #8 in 3.1.1 is satisfied. (Such a T exists according to #8 in B.3.) The parameters $T > 0$, $\mu > 0$ and the interval $\Omega' \subset (0,1)$ will be kept fixed in all what follows. We also assume that $g := g_{\Omega'}$ is a smooth function satisfying $\operatorname{supp}(g) \subset \Omega'$ and $0 \le g(x) \le 1$.[3]

2.1 Linearization of Ψ

We write Ψ as

$$\Psi(y_0, y_1, u) = \Phi(y_0, u) - y_1,$$

where $\Phi : H_0^1(0,1) \times C([0,T]; H_0^1(0,1)) \to H_0^1(0,1)$ is the map $\Phi(y_0, u) := y(T; y_0, u)$. To see that the map Φ (and hence Ψ) is well-defined, we need to know that the initial value problem (1a)-(1c) has a unique solution $y \in C([0,T]; H_0^1(0,1))$ for any choice of data $y_0 \in H_0^1(0,1)$ and $u \in C([0,T]; H_0^1(0,1))$. This is known for the homogeneous NLS (i.e. $u \equiv 0$ in (1a)) in 1D; cf. [3, Corollary 3.5.2.]. It is fairly easy to convince oneself that NLS with an additional inhomogeneity $\tilde{u} \in C([0,T]; H_0^1(0,1))$ (which is given by $\tilde{u}(t,x) = g(x)u(t,x)$ in our case) can be treated with the same methods as the the homogeneous equation. It is also not difficult to verify that the map Φ (and, by extension, Ψ) is continuous and Fréchet differentiable. We omit the technical details. Note that

$$\begin{aligned}
&\text{(4a)} & \Phi(\phi_\mu, 0) &= \varphi_\mu(T) = e^{i\mu T}\phi_\mu \quad \Rightarrow \quad \Psi(\phi_\mu, e^{i\mu T}\phi_\mu, 0) = 0 \\
&\text{(4b)} & \partial_u \Psi(\phi_\mu, e^{i\mu T}\phi_\mu, 0) \cdot h &= \partial_u \Phi(\phi_\mu, 0) \cdot h \quad (\forall h \in C([0,T]; H_0^1(0,1))).
\end{aligned}$$

Moreover, the derivative $\partial_u \Phi(\phi_\mu, 0) : C([0,T]; H_0^1(0,1)) \to H_0^1(0,1)$ of Φ at the point $(\phi_\mu, 0)$ is given by

$$\partial_u \Phi(\phi_\mu, 0) \cdot h = z(T; h),$$

where $t \mapsto z(t) = z(t; h)$ is the solution of IBVP

$$\begin{aligned}
&\text{(5a)} & iz_t &= -z_{xx} + f(|\varphi_\mu|^2)z + 2f'(|\varphi_\mu|^2)\operatorname{Re}(\varphi_\mu \bar{z})\varphi + g(x)h(t,x) \\
&\text{(5b)} & z(t,0) &= z(t,1) = 0 \\
&\text{(5c)} & z(0,x) &\equiv 0
\end{aligned}$$

[3] This assumption can be made without loss of generality. To see this, assume that Theorem 1 is proved for this case. If $g_{\Omega'}$ is the "actual" (non-smooth) indicator function for the interval Ω', choose an open subinterval $\tilde{\Omega} \subset \Omega'$ and a smooth function $\tilde{g}$ with $\operatorname{supp}(\tilde{g}) \subset \tilde{\Omega}$ and $0 \le \tilde{g}(x) \le 1$. Then, by Theorem 1, there will be a control $\tilde{u}$ that solves the control problem (1a)-(1d) with Ω' replaced by $\tilde{\Omega}$. Now $u(t,x) := \tilde{g}(x)\tilde{u}(t,x)$ will be a suitable control for the original problem.

As usual, the time dependence of the term involving Re(...) is eliminated by the transformation

$$\tilde{z}(t) := e^{-i\mu t} z(t), \quad \tilde{h}(t,x) := e^{-i\mu t} h(t,x),$$

which, for $f(s) = -s$ (cubic focusing nonlinearity), gives the IBVP

$$i\tilde{z}_t = -\tilde{z}_{xx} + \mu\tilde{z} - \phi_\mu^2\tilde{z} - 2\phi_\mu^2\mathrm{Re}(\tilde{z}) + g(x)\tilde{h}(t,x) \tag{6a}$$

$$\tilde{z}(t,0) = \tilde{z}(t,1) = 0 \tag{6b}$$

$$\tilde{z}(0,x) \equiv 0 \tag{6c}$$

We will work with the real (2×2)-system arising from (6a)-(6c) by decomposition in real and imaginary parts. (We will drop the μ subscripts whenever there is no danger of ambiguity.) Consider the matrix operator

$$L := \begin{pmatrix} 0 & -\Delta + \mu - \phi^2(x) \\ \Delta - \mu + 3\phi^2(x) & 0 \end{pmatrix} =: \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}; \tag{7}$$

this operator is to be understood as an operator in the Hilbert space $H^{-1} \times H^{-1} = [H_0^1(0,1)]^* \times [H_0^1(0,1)]^*$ with domain $H_0^1(0,1) \times H_0^1(0,1) = [H_0^1(0,1)]^2$. Then eq. (6a) takes the form

$$Z_t = LZ + g(x)\begin{pmatrix} \mathrm{Im}(\tilde{h}(t,x)) \\ -\mathrm{Re}(\tilde{h}(t,x)) \end{pmatrix}, \tag{8}$$

where

$$Z(t,x) = \begin{pmatrix} \mathrm{Re}(\tilde{z}(t,x)) \\ \mathrm{Im}(\tilde{z}(t,x)) \end{pmatrix}.$$

The corresponding linearized control problem to be considered therefore reads.

$$Z_t = LZ + g(x)H(t,x) \tag{9a}$$

$$Z(t,0) = Z(t,1) = \binom{0}{0} \tag{9b}$$

$$Z(0,x) = \binom{0}{0} \tag{9c}$$

$$Z(T,x) = Z_1(x) \tag{9d}$$

Our main objective is to prove the following controllability result for this linear system.

Theorem 2 *The control problem (9a)-(9d) has a solution $H = H(t,x)$, i.e. for any $Z_1 \in [H_0^1(0,1)]^2$ there exists a control function $H \in C([0,T]; [H_0^1(0,1)]^2)$ such*

that the solution $Z \in C([0,T];[H_0^1(0,1)]^2)$ of (9a)-(9c) satisfies (9d). Moreover, the control H satisfies the estimate

$$\sup_{0 \le t \le T} \|H(t)\|_{H^1} \le C \|Z_1\|_{H^1}. \tag{10}$$

Before proceeding with the proof of this theorem, we describe how Theorem 1 follows from it.

2.2 Proof of Theorem 1

As mentioned in the introduction, our plan is to apply the IFT (see, e.g. [15, Theorem 4.B]) to the map $\Psi : [H_0^1(0,1)]^2 \times C([0,T];H_0^1(0,1)) \to H_0^1(0,1)$, $\Psi(y_0, y_1, u) := \Phi(y_0, u) - y_1$. Since

$$e^{-i\mu T} \partial_u \Phi(\phi, 0) \cdot e^{i\mu T} \tilde{h} = z(T; \phi, \tilde{h}). \tag{11}$$

the map $\partial_u \Phi(\phi, 0)$ is onto iff the equation (6a)-(6c) is exactly controllable, which, in turn, is the case precisely if the equation (9a)-(9c) is exactly controllable. Thus, by Th. refP, $\partial_u \Phi(\phi, 0)$ (and hence $\partial_u \Psi(\phi, e^{i\mu^2 T}, 0)$) is onto. Moreover,

$$\|\partial_u \Phi(\phi, 0) \cdot h\| = \|z(T)\|_{H^1} = \|Z(T)\|_{H^1} \ge C\|H\|_{H^1} = C\|h\|_{H^1},$$

which shows that the map $\partial_u \Phi(\phi, 0)$ is one-to-one as well and that its inverse is bounded. Thus, by the IFT, there exist neighbourhoods $Y_0 \times Y_1 \subset [H_0^1(0,1)]^2$ and $U \subset C([0,T];H^1(0,1))$ of $(\phi, e^{i\mu T}\phi)$ and 0, respectively, such that

$$\forall (y_0, y_1) \in Y_0 \times Y_1 \, \exists! u = u(y_0, y_1) \in U : \quad \Psi(y_0, y_1, u) = 0,$$

which concludes the proof of the theorem (assuming that Th. 2 has been proven), since $y_1 \in H_0^1(0,1)$ is a reachable state iff there exists a function $u \in C([0,T];H^1(0,1))$ such that $\Psi(y_0, y_1, u) = \Phi(y_0, u) - y_1 = 0$. $\square$

3 Exact controllability of linearized NLS

The purpose of this section is to prove Theorem 2. The proof will employ the *Hilbert Uniqueness Method (HUM)* due to J.-L. Lions [9]. We will be using the decomposition $L = L^{(0)} + L^{(1)}$ of L, given by

$$L^{(0)} := \begin{pmatrix} 0 & -\Delta + \mu \\ \Delta - \mu & 0 \end{pmatrix} \quad \text{and} \quad L^{(1)} := \begin{pmatrix} 0 & -\phi^2(x) \\ 3\phi^2(x) & 0 \end{pmatrix}.$$

We will also need the adjoint operator

$$L^* = \begin{pmatrix} 0 & \Delta - \mu + 3\phi^2(x) \\ -\Delta + \mu - \phi^2(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -L_+ \\ L_- & 0 \end{pmatrix}. \tag{12}$$

The set-up is similar to the one in [6]. We use capital letters to remind ourselves that we are dealing with two-component functions. All equations below are to be understood subject to zero boundary conditions (i.e. are to be interpreted in the Hilbert space $[H_0^1(0,1)]^2$). The equation

$$\begin{cases} W_t = LW + g \cdot V \\ W(T,x) = Z_1(x) \end{cases}, \tag{13}$$

is decomposed into two "semi-homogeneous" equations:

$$W_t^{(1)} = LW^{(1)}, \quad W^{(1)}(T,x) = Z_1(x) \tag{14a}$$

$$W_t^{(2)} = LW^{(2)} + g \cdot V, \quad W^{(2)}(T,x) = 0, \quad \text{where} \tag{14b}$$

$$W = W^{(1)} + W^{(2)} \tag{14c}$$

The "HUM operator" is defined as

$$SV_0 := -W^{(2)}(0) \tag{15}$$

where $W^{(2)}$ is the solution of (14b) with V given by

$$V_t = -L^*V, \quad V(0) = V_0. \tag{16}$$

If S possesses a (bounded) inverse (in the space $[H_0^1(0,1)]^2$), then the equation

$$SV_0 = W^{(1)}(0)$$

has a unique solution V_0, and $W = W^{(1)} + W^{(2)}$ will satisfy (13) with $W(0) = 0$. So setting

$$Z = W, \quad H = V$$

will solve the control problem (9a)-(9d). Moreover, the estimate (10) will follow from the apriori estimates (27a) and (27b) listed in Section 3.2.1. The crux of the HUM, therefore, consists in showing that S has a bounded inverse. This is done by showing that S satisfies an "observability estimate" of the form

$$\langle SV_0, V_0 \rangle \geq C\|V_0\|^2 \qquad (\forall V_0 \in [H_0^1(0,1)]^2)$$

(w.r.t. the appropriate inner product and norm), which, by the Lax-Milgram Theorem, implies that S is an isomorphism of $[H_0^1(0,1)]^2$.

3.1 The L^2–observability estimate

The space $[L^2(0,1)]^2 = L((0,1);\mathbb{C}^2)$ is equipped with the standard inner product $\langle .,. \rangle$, given by

$$\begin{aligned}
\langle V^{(1)}, V^{(2)} \rangle &= \left\langle \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}, \begin{pmatrix} u^{(2)} \\ v^{(2)} \end{pmatrix} \right\rangle \\
&= \langle u^{(1)}, u^{(2)} \rangle_{L^2((0,1),\mathbb{C})} + \langle v^{(1)}, v^{(2)} \rangle_{L^2((0,1),\mathbb{C})} \\
&= \int_0^1 u^{(1)}(x)\overline{u^{(2)}(x)}dx + \int_0^1 v^{(1)}(x)\overline{v^{(2)}(x)}dx;
\end{aligned}$$

the corresponding norm is $\|V\| = \sqrt{\langle V, V \rangle}$. The objective is to show that there exists a constant C_{HUM} such that

$$\langle SV_0, V_0 \rangle \ \geq C_{HUM}\|V_0\|^2 \qquad (\forall V_0 \in [L^2(0,1)]^2) \tag{17}$$

The proof of (17) will make use of the following properties of the operator L^*.

3.1.1 Spectral decomposition of $-L^*$ and solutions to (16)

The justification of the properties listed in this section will be deferred to the appendix (see B.3).

1. the spectrum of $-L^*$ consists of eigenvalues only
2. all non-zero eigenvalues are purely imaginary
3. all but finitely many non-zero eigenvalues are simple; there are no generalized eigenvectors associated with non-zero eigenvalues and each eigenspace is at most two-dimensional.
   Remark. In the sequel, we will for convenience assume that *all* non-zero eigenvalues are simple[4]. All subsequent arguments (in particular, #5 below) can easily be adapted to accommodate additional (linearly independent) eigenvectors which potentially occur for a finite number of eigenvalues.
4. the multiplicity of the eigenvalue zero is 2; an eigenvector and a generalized eigenvector, V_1 and W_1, satisfying

$$-L^*V_1 = \binom{0}{0} \quad \text{and} \quad -L^*W_1 = V_1, \tag{18}$$

[4]There is persuasive numerical evidence to support this assumption.

are given by

$$V_1 = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \quad \text{and} \quad W_1 = \begin{pmatrix} 0 \\ \partial_\mu \phi \end{pmatrix} \tag{19}$$

(in particular, $V_1, W_1 \in \mathbb{R}^2$). Moreover, these vectors form a basis of the generalized null space.

5. Let $V_2, V_3, \ldots, \bar{V}_2, \bar{V}_3, \ldots$ be the eigenvectors corresponding to the simple non-zero eigenvalues (see Remark in #3 above) $\lambda_2, \lambda_3, \ldots, \bar{\lambda}_2, \bar{\lambda}_3, \ldots$. Then $V_1, W_1, V_2, V_3, \ldots, \bar{V}_2, \bar{V}_3, \ldots$ is a Schauder basis for $[L^2(0,1)]^2$ as well as a Bessel sequence, i.e. for any $V \in [L^2(0,1)]^2$, there is a unique representation

$$V = c_1 V_1 + d_1 W_1 + \sum_{n \geq 2} [c_n V_n + \hat{c}_n \bar{V}_n] \qquad ((c_n, \hat{c}_n) \in \ell^2) \tag{20}$$

and there exists a constant $B > 0$ (independent of V) such that

$$|c_1|^2 + |d_1|^2 + \sum_{n \geq 2} [|c_n|^2 + |\hat{c}_n|^2] \geq B \|V\|^2 \tag{21}$$

6. The (generalized) eigenfunctions $V_1, W_1, V_2, V_3, \ldots$ satisfy the uniform bound

$$m_V := \min \left\{ \inf_{n \geq 1} \int_\Omega g(x) |V_n(x)|^2 dx, \int_\Omega g(x) |W_1(x)|^2 dx \right\} > 0. \tag{22}$$

7. Let $V_0 \in [L^2(0,1)]^2$ be real, $V_0 = c_1 V_1 + d_1 W_1 + \sum_{n \geq 2} [c_n V_n + \bar{c}_n \bar{V}_n] = c_1 V_1 + d_1 W_1 + 2\mathrm{Re} \sum_{n \geq 2} c_n V_n$ and $\lambda_n = i\beta_n$ ($\beta_n \in \mathbb{R}$) . Then

$$\begin{aligned} V(t) &= c_1 V_1 + t d_1 W_1 + \sum_{n \geq 2} [c_n e^{i\beta_n t} V_n + \bar{c}_n e^{-i\beta_n t} \bar{V}_n] \\ &= c_1 V_1 + t d_1 W_1 + 2\mathrm{Re} \sum_{n \geq 2} c_n e^{i\beta_n t} V_n \end{aligned} \tag{23}$$

is the unique solution of (16).

8. The sequence

$$1, \quad t, \quad e^{i\beta_n t}, \quad e^{-i\beta_n t} \qquad (n \geq 2)$$

is a Riesz-Fischer sequence in $L^2(0,T;\mathbb{C})$, i.e., there exists a constant $A>0$ such that, for any ℓ^2–sequence $a_1, b_1, (a_n, \hat{a}_n)_{n\geq 2} \subset \mathbb{C}$,

$$\begin{aligned}\int_0^T \Big| a_1 + b_1 t + \sum_{n\geq 2} [a_n e^{i\beta_n t} + \hat{a}_n e^{-i\beta_n t}] \Big|^2 dt \\ \geq A\Big(|a_1|^2 + |b_1|^2 + \sum_{n\geq 2} [|a_n|^2 + |\hat{a}_n|^2] \Big)\end{aligned} \tag{24}$$

3.1.2 Proof of (17)

First we show that

$$\langle SV_0, V_0 \rangle \;=\; \int_0^T \langle gV, V\rangle dt \tag{25}$$

Proof of (25). Let

$$\tilde{S}(t) := \langle W^{(2)}(t), V(t)\rangle.$$

Then

$$\begin{aligned}\frac{d}{dt}\tilde{S}(t) \quad &= \quad \langle W_t^{(2)}, V\rangle \;+\; \langle W^{(2)}, V_t\rangle \\ &\overset{(14b),(12)}{=} \langle LW^{(2)}, V\rangle \;+\; \langle gV, V\rangle \;-\; \langle W^{(2)}, L^* V\rangle \;=\; \langle gV, V\rangle\end{aligned}$$

and so

$$\begin{aligned}\langle SV_0, V_0\rangle \;&=\; -\langle W^{(2)}(0), V(0)\rangle \;=\; \langle \underbrace{W^{(2)}(T),}_{=0} V(T)\rangle \;-\; \langle W^{(2)}(0), V(0)\rangle \\ &=\; \tilde{S}(T) - \tilde{S}(0) = \int_0^T \frac{d}{dt}\tilde{S}(t) dt = \int_0^T \langle gV, V\rangle dt,\end{aligned}$$

which completes the proof of (25). □

Now (17) may be verified using standard arguments. The solution $V(t)$ of (27a) has the representation

$$V(t,x) = c_1 V_1(x) + t d_1 W_1(x) + \sum_{n\geq 2} [c_n e^{i\beta_n t} V_n(x) + \bar{c}_n e^{-i\beta_n t} \bar{V}_n(x)]$$

(cf. (23)). We are going to apply (24) with

$$a_1 = c_1 V_1(x), \quad b_1 = d_1 W_1(x), \quad a_n = c_n V_n(x), \quad \hat{a}_n = \bar{c}_n \bar{V}_n(x);$$

the result is

$$\begin{aligned}
\langle SV_0, V_0\rangle &\overset{(25)}{=} \int_0^T \langle gV, V\rangle dt = \int_\Omega g(x) \int_0^T |V(t,x)|^2 dt dx \\
&\overset{(24)}{\geq} A \int_\Omega g(x)\Big(|c_1|^2|V_1(x)|^2 + |d_1|^2|W_1(x)|^2 + 2\sum_{n\geq 2}|c_n|^2|V_n(x)|^2\Big)dx \\
&\overset{(22)}{\geq} A\, m_V\Big(|c_1| + |d_1|^2 + 2\sum_{n\geq 2}|c_n|^2\Big) \overset{(21)}{\geq} A\, B\, m_V \|V_0\|^2,
\end{aligned}$$

which completes the proof of (17). □

3.2 The H^1-observability estimate

Now that the L^2–observability estimate (17) has been established, it is sufficient to show that there exist constants C_1 and C_2 such that

$$\langle (SV_0)_x, V_{0x}\rangle \;\geq C_1\|V_{0x}\|^2 - C_2\|V_0\|^2 \tag{26}$$

Indeed this will imply

$$\begin{aligned}
&\langle (SV_0)_x, V_{0x}\rangle \;\geq C_1\|V_{0x}\|^2 - \frac{C_2}{C_{HUM}}\langle SV_0, V_0\rangle \\
\Rightarrow \quad &\frac{C_2}{C_{HUM}}\,\langle SV_0, V_0\rangle \;+\; \langle (SV_0)_x, V_{0x}\rangle \;\geq C_1\|V_{0x}\|^2.
\end{aligned}$$

The left-hand side is equivalent to the "natural" H^1 inner product

$$\langle .,.\rangle + \langle (.)_x, (.)_x\rangle.$$

Before presenting the proof of (26) we list the apriori estimates required.

3.2.1 Apriori estimates

We are going to need apriori estimates for the various functions involved. The proofs are standard fare and will be omitted.

$$\begin{aligned}
&\text{(27a)} \quad & \|V(t)\| &\leq C\|V_0\| \\
&\text{(27b)} \quad & \|V_x(t)\| &\leq C(\|V_0\| + \|V_{0x}\|) \\
&\text{(27c)} \quad & \|W^{(2)}(t)\| &\leq C\|V_0\| \\
&\text{(27d)} \quad & \|W^{(2)}_x(t)\| &\leq C(\|V_0\| + \|V_{0x}\|)
\end{aligned}$$

We will also need the equations for V_x and $W_x^{(2)}$, which are given by

$$
\begin{aligned}
&\text{(28a)} \qquad & {W_x^{(2)}}_t &= LW_x^{(2)} + L_x^{(1)}W^{(2)} + g_xV + gV_x, \quad W_x^{(2)}(T) = 0 \\
&\text{(28b)} \qquad & V_{xt} &= -L^*V_x - [L_x^{(1)}]^*V, \quad V_x(0) = V_{0x}\cdot,
\end{aligned}
$$

3.2.2 Proof of (26)

We first prove an "H^1 analogy" of the identity (25):

$$
\text{(29)} \qquad \left\{ \begin{aligned} \langle (SV_0)_x, V_{0x}\rangle &= \int_0^T \Big[\langle L_x^{(1)}W^{(2)}, V_x\rangle - \langle L_x^{(1)}W_x^{(2)}, V\rangle \Big]\, dt \\ &\quad + \int_0^T \langle g_xV, V_x\rangle\, dt + \int_0^T \langle gV_x, V_x\rangle dt \end{aligned} \right.
$$

Proof of (29). Let

$$
\tilde{\tilde{S}}(t) := \langle W_x^{(2)}(t), V_x(t)\rangle.
$$

Then

$$
\begin{aligned}
\frac{d}{dt}\tilde{\tilde{S}}(t) &= \langle {W_x^{(2)}}_t, V_x\rangle + \langle W_x^{(2)}, V_{xt}\rangle \\
&= \langle LW_x^{(2)}, V_x\rangle + \langle L_x^{(1)}W^{(2)}, V_x\rangle + \langle g_xV, V_x\rangle + \langle gV_x, V_x\rangle \\
&\quad - \langle LW_x^{(2)}, V_x\rangle - \langle L_x^{(1)}W_x^{(2)}, V\rangle \\
&= \langle L_x^{(1)}W^{(2)}, V_x\rangle - \langle L_x^{(1)}W_x^{(2)}, V\rangle + \langle g_xV, V_x\rangle + \langle gV_x, V_x\rangle,
\end{aligned}
$$

which implies (29). □

Using the abbreviations

$$
\begin{aligned}
&\text{(30a)} \qquad & I_1 &:= \int_0^T \Big[\langle L_x^{(1)}W^{(2)}, V_x\rangle - \langle L_x^{(1)}W_x^{(2)}, V\rangle \Big]\, dt \\
&\text{(30b)} \qquad & I_2 &:= \int_0^T \langle g_xV, V_x\rangle dt \\
&\text{(30c)} \qquad & I_3 &:= \int_0^T \langle gV_x, V_x\rangle dt
\end{aligned}
$$

we estimate

$$
\text{(31)} \qquad \langle (SV_0)_x, V_{0x}\rangle \overset{(29)}{\geq} I_3 - |I_1| - |I_2|.
$$

Now,

$$
\begin{aligned}
|I_1| \quad & \leq \quad 3\|(\phi^2)_x\|_\infty \int_0^T (\|W^{(2)}\| \cdot \|V_x\| + \|W_x^{(2)}\| \cdot \|V\|)dt \\
& \overset{(27\text{a})-(27\text{d})}{\leq} \quad C\|V_0\| \cdot (\|V_0\| + \|V_{0x}\|) \\
& \overset{Young}{\leq} \quad \varepsilon\|V_{0x}\|^2 + C_\varepsilon\|V_0\|^2. \qquad (32)
\end{aligned}
$$

The estimate for I_2 is even simpler.

$$
\begin{aligned}
|I_2| \quad & \leq \quad \|g_x\|_\infty \int_0^T \|V\| \cdot \|V_x\|dt \overset{(27\text{a}),(27\text{b})}{\leq} C\|V_0\| \cdot (\|V_0\| + \|V_{0x}\|) \\
& \overset{Young}{\leq} \quad \varepsilon\|V_{0x}\|^2 + C_\varepsilon\|V_0\|^2 \qquad (33)
\end{aligned}
$$

Finally, we need to estimate I_3 from below. To do this, we write the solution of (28b) in Duhamel form

$$
V_x(t) = e^{-L^*t}V_{0x} - \int_0^t e^{-L^*(t-s)}[L_x^{(1)}]^*V(s)ds, \qquad (34)
$$

where e^{-L^*t} denotes the semi-group corresponding to the equation (16). If we set

$$
V^{(0)}(t) := e^{-L^*t}V_{0x},
$$

we get from (17) and (25) that

$$
\int_0^T \int_\Omega g(x)|V^{(0)}(t,x)|^2 dxdt = \int_0^T \langle gV^{(0)}, V^{(0)}\rangle dt \geq C_{HUM}\|V_{0x}\|^2 \qquad (35)
$$

(here we mean by $|.|$ the norm of $\mathbb{C}^2$, i.e. $\left|\binom{u}{v}\right|^2 = |u|^2 + |v|^2$.) Now,

$$
\begin{aligned}
|I_3| \quad & \overset{(34)}{\geq} \quad \int_0^T \int_\Omega g(x)|V^{(0)}(t,x)|^2 dxdt - \cdots \\
& \qquad - \int_0^T \int_\Omega g(x) \left| \int_0^t e^{-L^*(t-s)}[L_x^{(1)}]^*V(s,x)ds \right|^2 dxdt \\
& \overset{(35)}{\geq} \quad C_{HUM}\|V_{0x}\|^2 - \int_0^T \int_\Omega g(x) \left| \int_0^t e^{-L^*(t-s)}[L_x^{(1)}]^*V(s,x)ds \right|^2 dxdt
\end{aligned}
$$

To finish the proof, we need to estimate the integral term.

$$\begin{aligned}
\int_0^T \int_\Omega g(x) \left|...\right|^2 dxdt &\leq \int_0^T \int_0^T \int_\Omega \left|e^{-L^*(t-s)}[L_x^{(1)}]^* V(s,x)\right|^2 dxdsdt \\
&= \int_0^T \int_0^T \left\|e^{-L^*(t-s)}[L_x^{(1)}]^* V(s,.)\right\|^2 dsdt \\
&\overset{(27a)}{\leq} C\int_0^T \int_0^T \left\|[L_x^{(1)}]^* V(s,.)\right\|^2 dsdt \\
&\leq C\|(\phi^2)_x\|^2 \int_0^T \|V(s)\|^2 ds \overset{(27a)}{\leq} C\|V_0\|^2
\end{aligned}$$

Thus,

$$I_3 \geq C_{HUM}\|V_{0x}\|^2 - C\|V_0\|^2 \tag{36}$$

and so

$$\langle (SV_0)_x, V_{0x}\rangle \overset{(31)}{\geq} I_3 - |I_1| - |I_2| \overset{(32),(33),(36)}{\geq} (C_{HUM} - 2\varepsilon)\|V_{0x}\|^2 - C_\varepsilon\|V_0\|^2,$$

which will conclude the proof of (26) if ε is chosen sufficiently small. □

4 Concluding remarks

There are a number of modifications/generalizations of the control problem (1a)-(1d) which are of interest. Those include

- space dimension > 1;
- other boundary conditions such as periodic boundary conditions;
- the ground state ϕ_μ in the assumption of Theorem 1 may be replaced with some excited state (see A.1);
- zero (or "box") boundary conditions may be interpreted as an infinite potential well, which one may want to replace with other, typically confining, potentials, such as the harmonic-oscillator potential (in this case, one would reasonably set $\Omega = (-\infty, \infty)$.)

Any of these modifications will obviously change some of the spectral properties in 3.1.1. To see whether a controllability result as in Theorem 1 can be proved under these modified assumptions as well, the ramifications for the application of HUM will have to be carefully examined.

Appendix

A Bound states

We define *bound states* as real solutions of the BVP (2a),(2b).

A.1 Bound states in terms of elliptic functions

It is well-known that explicit formulas for the solutions of (2a),(2b) are available in terms of Jacobian elliptic functions; see, e.g. [2]. If $j \in \{0, 1, 2, \ldots\}$, then $\phi_j(x)$ will denote the (real-valued) solution of (2a),(2b) which possesses precisely j zeros ("nodes") within the interval $(0, 1)$. The node-less solution $\phi := \phi_0$ is referred to as the *ground state*; the solutions ϕ_j $(j \geq 1)$ with one or multiple nodes are called *excited states.* To find an explicit solution formula for ϕ_j, we first solve the equation

$$\mu = 4(j+1)^2(2k^2-1)K(k)^2 \tag{37}$$

for k, where $K(k)$ denotes the complete elliptic integral of the first kind (see, e.g. [1]). Note that, since $K(k)$ is a strictly increasing continuous function of $k \in [0, 1)$ satisfying $\lim_{k\to 1^-} K(k) = \infty$, equation (37) has exactly one solution $k = k_j(\mu)$ for any choice of parameters $\mu \geq 0$ and $j \in \{0, 1, 2, \ldots\}$. Moreover, the function $k_j : [0, \infty) \to [\frac{1}{\sqrt{2}}, 1)$ is continuous and strictly increasing as well, and satisfies $\lim_{s\to\infty} k_j(s) = 1$. Now the solution ϕ_j of (2a),(2b) is given by $(k = k_j(\mu))$

$$\begin{aligned} \phi_j(x) &= \frac{\sqrt{2\mu}\,k}{\sqrt{2k^2-1}} \operatorname{cn}\left(\frac{\sqrt{\mu}(x-\frac{1}{2})}{\sqrt{2k^2-1}} + [j]_2K(k), k\right) \\ &= 2\sqrt{2}(j+1)kK(k) \operatorname{cn}\Big(2(j+1)K(k)(x-\tfrac{1}{2}) + [j]_2K(k), k\Big), \end{aligned} \tag{38}$$

where $[j]_2 := j \bmod 2$. The following properties are readily proved.

Lemma 3 *Let $\psi \in H_0^1(0,1)$ be a non-trivial weak solution of (2a), (2b), i.e.*

$$\int_0^1 [\psi_x v_x + \mu\psi v - \psi^3 v]\, dx = 0 \qquad (\forall v \in H_0^1(0,1)).$$

Then

(i) $\psi \in C^\infty(0,1) \cap C[0,1]$, *i.e.* ψ *is a classical solution of (2a), (2b).*

(ii) There exists $j \in \{0,1,2,\ldots\}$ *such that* ψ *has exactly* j *zeros in* $(0,1)$.

(iii) $\psi(x) = \phi_j(x)$, *where* ϕ_j *is given by (38).*

We also need the following convexity property.

Lemma 4 *The function* $\mu \mapsto \|\phi\|_2^2$ *is an increasing function of* μ.

Proof. This follows from the identity

$$\|\phi\|_2^2 = 4k^2 K(k) \int_{-K(k)}^{K(k)} \mathrm{cn}^2(y,k) dy \tag{39}$$

and the fact that $k = k(\mu)$ is a strictly increasing function of μ. □

A.2 Variational description of the ground state

There are various variational descriptions of the ground state in the whole-space case; see e.g. [12, 4.2] and [3, 8.1]; in the zero-boundary case we find the formulation presented in Lemma 5 below to be a useful one. Let μ_1 be the smallest eigenvalue of the 1D Laplacian $-\frac{d^2}{dx^2}$ on $(0,1)$ with zero boundary conditions, i.e. $\mu_1 = \pi^2$.

Lemma 5 *Let* $\mu > -\mu_1$. *Then there exists a positive minimizer*

$$w \in \{u \in H_0^1(0,1) \mid \int_0^1 u^4 dx = 1\} =: M$$

for the constrained minimization problem

$$\inf_{u \in M} \frac{1}{2} \int_0^1 [{u_x}^2 + \mu u^2] dx,$$

and the unique positive solution ϕ *of (2a),(2b) is given by* $\phi(x) = \lambda^{1/2} w(x)$, *for some suitable* $\lambda > 0$ *(Lagrange multiplier).*

Proof. See e.g. [11, Theorem 2.1 and proof]. □

B Spectral properties of $-L^*$

B.1 Properties of L_+ and L_-

The operators

$$L_-u = \left[-\tfrac{d^2}{dx^2} + \mu - \phi^2(x)\right]u, \quad u(0) = u(1) = 0 \tag{40a}$$

$$L_+u = \left[-\tfrac{d^2}{dx^2} + \mu - 3\phi^2(x)\right]u, \quad u(0) = u(1) = 0 \tag{40b}$$

are regular Sturm-Liouville (SL) operators, so we can make use of the SL theory.

Lemma 6 *(i) $ker(L_-) = span\{\phi\}$*

(ii) $L_-|_{[span(\phi)]^\perp} > 0$

Proof. *(i)* Clearly, $L_-\phi = 0$, so ϕ is eigenfunction for L_- with eigenvalue $\lambda = 0$. Since all eigenspaces are one-dimensional, the assertion follows.

(ii). Since ϕ has no zeros in $(0, 1)$, it is an eigenfunction for the smallest eigenvalue; hence $\lambda_1 = 0$ and $\lambda_n > 0$ $(\forall n \geq 2)$, which implies assertion (ii). □

Remark. If ϕ is not the ground state but an excited state with $j \in \mathbb{N} \setminus \{0\}$ nodes, then the operator L_- will have exactly j negative eigenvalues.

Lemma 7 *The operator L_+ has exactly one negative eigenvalue and all the other eigenvalues are positive. In particular, $ker(L_+) = \{0\}$.*

Proof. The proof proceeds in four steps.
Step 1. L_+ possess *at least one* negative eigenvalue. This follows from

$$\langle L_+\phi, \phi\rangle = \langle L_-\phi, \phi\rangle - 2\|\phi\|_4^4 = -2\|\phi\|_4^4 < 0$$

and the minimax principle.
Step 2. $\langle L_+\eta, \eta\rangle \geq 0$ for all $\eta \in [\text{span}(\phi^3)]^\perp$. This is a slight adaptation of the arguments in [10, Section13]; we use similar notation. Let the functionals $J[u]$ and $W[u]$ be defined by

$$J[u] := \frac{1}{2}\int_0^1 [{u_x}^2 + \mu u^2]dx \quad \text{and} \quad W[u] := \frac{1}{4}\int_0^1 u^4 dx$$

and let $w \in M = \{u \in H_0^1(0, 1) \mid W[u] = 1\}$ be a positive minimizer of the constrained minimization problem $\inf_{u\in M} J[u]$. By Lemma 5, the ground state ϕ is

given by $\phi(x) = \lambda^{1/2} w(x)$ for some positive constant λ, which arises as a Lagrange multiplier. Now let $\eta \in [\mathrm{span}(\phi^3)]^\perp$; we write η in the form $\eta = \dot{w} := \frac{\partial}{\partial z}|_{z=0} w(.,z)$ where $z \mapsto w(.,z)$ is a smooth curve in $H_0^1(0,1)$ such that $W[w(.,z)] = 1$ (i.e. $w(.,z) \in M$) for all z, and $w(.,0) = w$. Since w is a minimizer, we have

$$0 = \frac{d}{dz}\Big|_{z=0} J[w(.,z)] = \int_0^1 [w_x \dot{w}_x + \mu w \dot{w}] dx \tag{41}$$

and

$$\begin{aligned} 0 \;&\leq\; \frac{d^2}{dz^2}\Big|_{z=0} J[w(.,z)] = \int_0^1 [\dot{w}_x^2 + w_x \ddot{w}_x + \mu(\dot{w}^2 + w\ddot{w})] dx \\ &=\; -\int_0^1 [(\dot{w}_{xx} - \mu\dot{w})\dot{w} + (w_{xx} - \mu w)\ddot{w}] dx \end{aligned} \tag{42}$$

where $\ddot{w} := \frac{\partial^2}{\partial z^2}|_{z=0} w(.,z)$ Moreover, from the constraint $W[w(.,z)] \equiv 1$ we get

$$0 = \frac{d}{dz}\Big|_{z=0} W[w(.,z)] = \int_0^1 w^3 \dot{w} dx \tag{43}$$

and

$$0 = \frac{d^2}{dz^2}\Big|_{z=0} W[w(.,z)] = \int_0^1 [3w^2\dot{w}^2 + w^3\ddot{w}] dx. \tag{44}$$

Also, the Lagrange-multiplier rule implies

$$\int_0^1 [w_x v_x + \mu w v]\, dx \;=\; \lambda \int_0^1 w^3 v\, dx \quad (\forall v \in H_0^1(0,1)) \tag{45}$$

$$\overset{v=\ddot{w}}{\Rightarrow} \quad \int_0^1 [-w_{xx} + \mu w]\ddot{w}\, dx \;=\; \lambda \int_0^1 w^3 \ddot{w} dx \overset{(44)}{=} -\lambda \int_0^1 3w^2\dot{w}^2 dx \tag{46}$$

Inserting this into (42) gives

$$\begin{aligned} 0 \quad &\leq \quad \int_0^1 [-\dot{w}_{xx} + \mu\dot{w} - \lambda 3w^2\dot{w}]\dot{w} dx \\ &\overset{\phi=\lambda^{1/2}w,\, \eta=\dot{w}}{=} \quad \int_0^1 [-\eta_{xx} + \mu\eta - 3\phi^2\eta]\eta\, dx = \langle L_+\eta, \eta\rangle \end{aligned}$$

(Note that choosing $v = w$ in (45) also yields $\int_0^1 [w_x^2 + \mu w^2]\, dx = \lambda \int_0^1 w^4\, dx$ implying that $\lambda > 0$, since $\mu > -\mu_1$.)

STEP 3. The second eigenvalue, λ_2, is non-negative. This can be shown by repeating word-by-word the proof in [10, page 58] if Ψ is defined as ϕ^3 and Π is interpreted as the orthogonal projection onto the subspace $[\mathrm{span}(\phi^3)]^\perp$.
STEP 4. $\lambda_2 > 0$. Assume that $\lambda_2 = 0$ and let v denote an eigenfunction for λ_2. By the symmetry of $\phi(x)$, we may assume w.l.o.g. that v is either odd (i.e. $v(1-x) = -v(x)$) or even (i.e. $v(1-x) = v(x)$). In the first case, we have $v(\frac{1}{2}) = 0$ and v coincides with a constant multiple of ϕ' by ODE uniqueness. However, this is impossible, since ϕ' does not satisfy zero boundary conditions. We are therefore left with the second case (v even). Since v is an eigenfunction for the second eigenvalue, it has precisely one zero in $(0,1)$ by standard SL theory. By symmetry this zero must occur at $x = \frac{1}{2}$, which is impossible as we saw above. $\square$

B.2 Eigenvalues and eigenfunctions for $n \to \infty$

The properties 5 and 6 in Section 3.1.1 are based on asymptotic ($n \to \infty$) formulas for the eigenvalues $\lambda_n, \bar{\lambda}_n$ and eigenfunctions $V_n(x)$, $\bar{V}_n(x)$ of $-L^*$, which are given in Lemmas 8 and 9 below. We are going to make use of the fact (proved in B.3 #2 below) that all non-zero eigenvalues are purely imaginary, i.e. we write

$$\lambda_n = i\beta_n, \quad \bar{\lambda}_n = -i\beta_n, \qquad \beta_n \in \mathbb{R},\, \beta_n > 0. \tag{47}$$

Moreover, it will be convenient to employ a similarity transformation [10, (12.15)]: Let

$$J := \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Then

$$-iL^* = J \cdot \left[\begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \phi^2 \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \right] \cdot J^{-1}. \tag{48}$$

and so

$$\mathrm{spec}(-L^*) = i\mathrm{spec}(M), \quad V_n = JW_n^+, \quad \bar{V}_n = JW_n^-$$

where $(\pm\beta_n, W_n^\pm)$ are the eigenpairs for the operator

$$M := \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \phi^2 \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} =: M^{(0)} + M^{(1)},$$

i.e.

$$MW_n^\pm = \pm\beta_n W_n^\pm. \tag{49}$$

Writing $\beta = \pm\beta_n$, $W = W_n^\pm = \binom{u}{v}$, the characteristic equation (49) is equivalent to the BVP

$$
\begin{aligned}
&\text{(50a)} \qquad & u'' - (\mu+\beta)u &= -\phi^2(2u-v), \qquad u(0)=u(1)=0 \\
&\text{(50b)} \qquad & v'' - (\mu-\beta)v &= \phi^2(u-2v), \qquad v(0)=v(1)=0.
\end{aligned}
$$

Note that

$$
W_n^+ = \begin{pmatrix} u \\ v \end{pmatrix} \quad \Longleftrightarrow \quad W_n^- = \begin{pmatrix} v \\ u \end{pmatrix}. \tag{51}
$$

Lemma 8 *(i) The operator M (and hence $-L^*$) is a spectral operator. More precisely, the collection of eigenvectors and generalized eigenvectors for M forms a Schauder basis for $L^2(0,1;\mathbb{C})$ and all eigenvalues with sufficiently large indices n are simple.*

(ii) There exists an index $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that

$$
|(n^2\pi^2 + \mu) - \beta_n| \leq C
$$

for all $n \geq n_0$.

Proof. The operator $M^{(0)}$ is self-adjoint; its eigenvalues and eigenfunctions are given by

$$
\beta_n^{(0)\pm} = \pm(n^2\pi^2+\mu), \quad W_n^{(0)+} = \sin(n\pi x)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad W_n^{(0)-} = \sin(n\pi x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

(where $M^{(0)}W_n^{(0)\pm} = \beta_n^{(0)\pm}W_n^{(0)\pm}$). Thus, the operator $M = M^{(0)} + M^{(1)}$ "is" a bounded perturbation of a self-adjoint operator whose spectrum consists of simple eigenvalues only. The assertion now follows from [8, Th. 4.15.a]; see also [5]. $\square$

Lemma 9 *There exist an index $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that $\beta_n > \mu$ and*

$$
\begin{aligned}
&\text{(52a)} \qquad & \left| W_n^+(x) - \sin\left(\sqrt{\beta_n - \mu}\; x\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| &\leq \frac{C}{n} \\
&\text{(52b)} \qquad & \left| W_n^-(x) - \sin\left(\sqrt{\beta_n - \mu}\; x\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| &\leq \frac{C}{n}
\end{aligned}
$$

for all $n \geq n_0$.

Remark. The assertion of the lemma may be expressed in a more intuitive, if slightly informal, manner by means of the asymptotic formulas

$$\begin{aligned} W_n^+(x) &= \sin(\omega_n x)\tbinom{0}{1} + \mathcal{O}(n^{-1}), \quad n \to \infty \\ W_n^-(x) &= \sin(\omega_n x)\tbinom{1}{0} + \mathcal{O}(n^{-1}), \quad n \to \infty. \end{aligned}$$

(uniformly in x and n) where $\omega_n := \sqrt{\beta_n - \mu} \sim \pi n$, as $n \to \infty$, by Lemma 8 (ii).

Proof. Clearly, by Lemma 8 (ii), $\lim_{n\to\infty} \beta_n = \infty$; so we may assume that $\beta_n - \mu > 0$. Let $\omega_n^\pm := \sqrt{\beta_n \pm \mu}$. Because of (51) it is sufficient to consider W_n^+; write $W_n^+(x) = \binom{u(x)}{v(x)}$. Viewing the R.H.S.'s of (50a),(50b) as inhomogeneities, we write the system as

$$u'' - [\omega_n^+]^2 u = f(x), \quad u(0) = u(1) = 0 \tag{53a}$$

$$v'' + [\omega_n^-]^2 v = g(x), \quad v(0) = v(1) = 0. \tag{53b}$$

where

$$f(x) = \phi^2(x)[v(x) - 2u(x)] \quad \text{and} \quad g(x) = \phi^2(x)[u(x) - 2v(x)]. \tag{53c}$$

Note that the homogeneous BVP associated with eq. (53b) has the solution $\sin(\omega_n^- x)$, while the homogeneous BVP associated with eq. (53a) does not have any non-trivial solution, which implies that there is a Green's function, $\Gamma_{\omega_n^+}(x, \xi)$, associated with eq. (53a). Utilizing this Green's function and the Duhamel Principle, solutions to (53a),(53b) may be written as

$$u(x) = \int_0^1 \Gamma_{\omega_n^+}(x,\xi) f(\xi) d\xi \tag{54a}$$

$$v(x) = c\sin(\omega_n^- x) + \frac{1}{\omega_n^-}\int_0^x \sin(\omega_n^-(x-\xi)) g(\xi) d\xi \quad (c \in \mathbb{R}) \tag{54b}$$

By the linearity of the system (50a),(50b), we may assume that $c = 1$. Thus,

$$u(x) = \int_0^1 \Gamma_{\omega_n^+}(x,\xi)\phi^2(\xi)[2u(\xi) - v(\xi)] d\xi \tag{55a}$$

$$v(x) = \sin(\omega_n^- x) + \frac{1}{\omega_n^-}\int_0^x \sin(\omega_n^-(x-\xi))\phi^2(\xi)[u(\xi) - 2v(\xi)] d\xi \tag{55b}$$

Standard calculations yield the explicit formula for the Green's function $\Gamma_{\omega_n^+}$, which is given by

$$(56)\begin{cases} \Gamma_{\omega_n^+}(x,\xi) = \dfrac{1}{4\omega_n^+ \sinh(\omega_n^+)} \Big\{ [\sinh(\omega_n^+\xi)e^{-\omega_n^+} - \sinh(\omega_n^+(1-\xi))]e^{\omega_n^+ x} \\ \qquad\qquad -[\sinh(\omega_n^+\xi)e^{\omega_n^+} - \sinh(\omega_n^+(1-\xi))]e^{-\omega_n^+ x} \Big\} \\ \qquad + \dfrac{\sinh(\omega_n^+|x-\xi|)}{2\omega_n^+} \end{cases}$$

The function $|\Gamma_{\omega_n^+}(x,\xi)|$ assumes its maximum on $[0,1]^2$ at the point $(x,\xi) = \left(\frac{1}{2},\frac{1}{2}\right)$ and its maximum value is given by

$$(57) \qquad \left|\Gamma_{\omega_n^+}\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right| = \frac{\sinh^2(\frac{\omega_n^+}{2})}{\omega_n^+ \sinh(\omega_n^+)} = \frac{\cosh(\omega_n^+)-1}{2\omega_n^+ \sinh(\omega_n^+)} = \mathcal{O}\left(\frac{1}{\omega_n^+}\right) \quad (\omega_n^+ \to \infty)$$

(uniformly in ω_n^+). Now it is a matter of routine estimates (combining the representation (55a),(55b) with the uniform estimate (57) and the property $\omega_n^\pm \sim n$, as $n \to \infty$) to verify the assertion of the lemma. □

B.3 Verification of the properties listed in Section 3.1.1

1. Lemma 8 (i).

2. Let $\lambda \in \mathbb{C} \setminus \{0\}$ be an eigenvalue for L. Then the argument on page 13 of [4] ("Claim") shows that λ^2 is real. We can then use the argument on pages 49 and 50 of [10] to show that λ itself is purely imaginary. This argument uses, besides the properties of L_+ listed in Lemma 7, the convexity property $\frac{d}{d\mu}\langle \phi, \phi\rangle = 2\langle \partial_\mu \phi, \phi\rangle > 0$ (Lemma 4).

3. The first assertion (all but finitely many eigenvalues are simple) was already mentioned in Lemma 8 (i).
   A proof of the second one (there are no generalized eigenvectors associated with non-zero eigenvalues) can be found in [10, pages 50-51]. The proof refers to the whole-space case, but carries over to the zero-boundary case if the domains of the various operators involved are modified suitably.
   The third assertion (each eigenspace is at most two-dimensional) may be seen as follows. Let $\lambda \in \mathbb{C}$ be an eigenvalue and $\mathrm{Eig}(\lambda)$ the corresponding eigenspace and consider the linear map $F : \mathrm{Eig}(\lambda) \to \mathbb{C}^2$, defined by $FV := V'(0)$. Using the left boundary condition ($V(0) = \binom{0}{0}$) and ODE uniqueness, is is easy to see that F is one-to-one. Thus $\dim(\mathrm{Eig}(\lambda)) = \dim(\mathrm{im}(F)) \leq 2$.

4. Let $\binom{u}{v}$ be an eigenfunction for L^* associated with the eigenvalue $\lambda = 0$. Then $L_- u = 0$ and $L_+ v = 0$. From Lemma 6 (i) and Lemma 7 we get $u = c\phi$ ($c \in \mathbb{C}$) and $v = 0$; hence
$$\begin{pmatrix} u \\ v \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} \phi \\ 0 \end{pmatrix} \right\}.$$
It is easy to see that $W_1 = \binom{0}{\partial_\mu \phi}$ satisfies $-L^* W_1 = V_1$; it is therefore a generalized eigenvector. We want to show that there cannot be another linearly independent generalized eigenvector. To prove this, let's assume that $\binom{u}{v}$ is such an "additional" generalized eigenvector, i.e.
$$-L^* \begin{pmatrix} u \\ v \end{pmatrix} = W_1 = \begin{pmatrix} 0 \\ \partial_\mu \phi \end{pmatrix},$$
since the eigenspace is one-dimensional. In particular,
$$L_- u = -\partial_\mu \phi. \tag{58}$$
By the Fredholm alternative, this implies
$$0 = 2\langle \phi, \partial_\mu \phi \rangle = \frac{d}{d\mu} \langle \phi, \phi \rangle \tag{59}$$
since ϕ is a non-trivial solution to $L_- \phi = 0$. But this contradicts Lemma 4. As a result, no solution u to (58) can exist, so there is no "additional" generalized eigenvector.

5. Basis property: Lemma 8 (i).

   The Bessel-sequence property follows from Lemma 9: Since it is obviously sufficient to establish that the sequence of (generalized) eigenvectors for M is a Bessel sequence, we will show that
$$\forall W = \binom{u}{v} \in [L^2(0,1)]^2 : \qquad (a_n^\pm)_{n \in \mathbb{N}} := (\langle W, W_n^\pm \rangle)_{n \geq \mathbb{N}} \in \ell^2. \tag{60}$$
Clearly, we may skip a finite number of terms in (60). For simplicity, we will also restrict ourselves to a_n^+. Let n_0 be an index such that
$$W_n^+(x) = \sin(\omega_n x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{R_n^+(x)}{n} \quad (\forall x \in [0,1],\ n \geq n_0),$$

where $\omega_n := \sqrt{\beta_n - \mu}$ and $R_n^+ : [0,1] \to \mathbb{C}^2$ are continuous functions satisfying $|R_n^+(x)| \le C$ uniformly in x and n (see Lemma 9). Now

$$a_n^+ = \int_0^1 v(x) \sin(\omega_n x) dx + \frac{\langle W, R_n^+ \rangle}{n} =: b_n^+ + c_n^+.$$

The sequence (c_n^+) is easily seen to be an ℓ^2-sequence:

$$|\langle W, R_n^+ \rangle| \le C \int_0^1 |W(x)| dx \le C\|W\| \quad \Rightarrow \quad |c_n^+| \le \frac{C\|W\|}{n} \in \ell^2.$$

To see that (b_n^+) is square-summable as well, we note that, by Lemma 8, the sequence $(\omega_n)_{n \ge n_0}$ has the asymptotics $\omega_n \sim \pi n$, as $n \to \infty$, and is therefore separated. By [13, Theorem 3.4], this implies that the exponential system $\{e^{i\omega_n x}\}_{n \ge n_0}$ forms a Bessel sequence in $L^2(0,1)$. Thus

$$b_{j,n}^+ := \int_0^1 v_j(x) e^{i\omega_n x} dx \in \ell^2 \qquad (j \in \{1,2\}),$$

where $v_1(x) := \mathrm{Re}(v(x))$ and $v_2(x) := \mathrm{Im}(v(x))$, and so

$$b_n^+ = \mathrm{Im}(b_{1,n}^+) + i\mathrm{Im}(b_{2,n}^+) \in \ell^2,$$

which completes the proof of $(a_n^+) \in \ell^2$.

6. This also follows from Lemma 9. Let $\Omega' = (a,b) \subset (0,1)$. Then we have

$$\int_{\Omega'} \sin^2(\omega x) dx \longrightarrow \frac{b-a}{2}, \quad \text{as } \omega \to \infty.$$

As above, we appeal to Lemma 9 to write W_n^+ in the form

$$W_n^+(x) = \sin(\omega_n x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{R_n^+(x)}{n} \quad (\forall x \in [0,1],\, n \ge n_0).$$

Now

$$\begin{aligned} \int_{\Omega'} |W_n^+(x)|^2 dx \;\; &\ge \;\; \int_{\Omega'} \sin^2(\omega_n x) dx - \frac{2}{n} \int_{\Omega'} \left| \sin(\omega_n x) \left\langle \tbinom{0}{1}, R_n^\pm(x) \right\rangle \right| dx - \cdots \\ &\qquad -\frac{1}{n^2} \int_{\Omega'} |R_n^\pm(x)|^2 dx \\ &\ge \;\; \int_{\Omega'} \sin^2(\omega_n x) dx - \frac{2C(b-a)}{n} - \frac{C^2(b-a)}{n^2} \longrightarrow \frac{b-a}{2}, \end{aligned}$$

as $n \to \infty$ (note that $\lim_{n \to \infty} \omega_n = \infty$). This concludes the proof.

7. clear
8. According to Lemma 8, we may write

$$\beta_n = \pi^2 n^2 + r_n \qquad (n \geq n_0),$$

where $|r_n| \leq C$ for some constant $C > 0$ (independent of n). It follows that

$$\beta_{n+1} - \beta_n = \pi^2[(n+1)^2 - n^2] + r_{n+1} - r_n = \pi^2[2n+1] + r_{n+1} - r_n \to \infty,$$

as $n \to \infty$, which, by [14, Corollary], implies that the sequence

$$\mathscr{E} := \{e^{-i\beta_n t}, 1, e^{i\beta_n t}\}_{n\geq 2}$$

is a Riesz-Fischer sequence in $L^2(0,T)$ for every $T > 0$. We will show that adding the function $f(t) = t$ to $\mathscr{E}$ will result in a Riesz-Fischer sequence in $L^2(0,T)$ as well, *if* $T > 0$ *is large enough.* For ease of notation, let's define

$$e_0(t) = 1, \quad e_{\pm m}(t) := e^{\pm i\beta_{m+1}t} \quad (m \geq 1).$$

The fact that the sequence $\mathscr{E}$ is a Riesz-Fischer means that there is a constant $A_{\mathscr{E}} > 0$ such that

$$\int_0^T \left| \sum_{n=-\infty}^{\infty} a_n e_n(t) \right|^2 dt \geq A_{\mathscr{E}} \sum_{n=-\infty}^{\infty} |a_n|^2 \qquad (\forall (a_n)_{n\in\mathbb{Z}} \in \ell^2) \tag{61}$$

Thus

$$\int_0^T \left| \sum_{n=-\infty}^{\infty} a_n e_n(t) \, + bf(t) \right|^2 dt \overset{(61)}{\geq} A_{\mathscr{E}}\|a\|_{\ell^2}^2 + \|f\|_{L^2(0,T)}^2 |b|^2 - 2|b| \, \|a\|_{\ell^2} \|F\|_{\ell^2}$$

where the sequence $(F_n)_{n\in\mathbb{Z}}$ is defined by

$$F_{\pm m} := \int_0^T e_{\pm m}(t) f(t) dt = \int_0^T e^{\pm i\beta_{m+1}t} f(t) dt \quad (m \geq 0).$$

(Since the sequence $(-\beta_m, \beta_m)_{m\in\mathbb{N}}$ is obviously separated, the sequence $\mathscr{E}$ is a Bessel sequence, which implies that $F \in \ell^2$. Since $f(t) = t$, the F_n's can also be computed explicitly to verify the square-summability of F.) Utilizing

the elementary Young's inequality, we can continue the estimation above to obtain

$$\int_0^T \left| \sum_{n=-\infty}^{\infty} a_n e_n(t) + b f(t) \right|^2 dt \;\; \geq \;\; \varepsilon \|a\|_{\ell^2}^2 + \left(\|f\|_{L^2(0,T)}^2 - \frac{\|F\|_{\ell^2}^2}{A_{\mathscr{E}} - \varepsilon} \right) |b|^2$$

(for $\varepsilon \in (0, A_{\mathscr{E}})$), which will yield the assertion provided that the condition

$$\|F\|_{\ell^2}^2 < (A_{\mathscr{E}} - \varepsilon) \|f\|_{L^2(0,T)}^2. \tag{62}$$

is satisfied. Clearly, $\|f\|_{L^2(0,T)}^2 = \frac{T^3}{3}$, since $f(t) = t$. Moreover, it is a matter of routine calculations to verify that $\|F\|_{\ell^2}^2 \leq C_f T^2$ for some T-independent constant C_f. Thus, condition (62) takes the form

$$C_f < \frac{T}{3}(A_{\mathscr{E}} - \varepsilon) \tag{63}$$

(note that the constant $A_{\mathscr{E}}$ depends on T, $A_{\mathscr{E}} = A_{\mathscr{E}}(T)$). Finally, it is easy to see that $A_{\mathscr{E}}$ can be chosen such that $A_{\mathscr{E}}(T_2) \geq A_{\mathscr{E}}(T_1)$ if $T_2 \geq T_1$, which implies that condition (63) (and hence (62)) can be fulfilled by choosing T large enough. This concludes the proof that the sequence $\mathscr{E} \cup \{f\}$ is a the Riesz-Fischer sequence in $L^2(0,T)$ if $T > 0$ is sufficiently large.

Acknowledgment. This research was supported by the Natural Sciences and Engineering Research Council of Canada through its Discovery Grant programme. The authors would like to thank R. Illner for valuable discussions.

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